

CROSSCAP SLIDES AND THE LEVEL 2 MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

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ABSTRACT. Crosscap slide is a homeomorphism of a nonorientable surface of genus at least 2, which was introduced under the name Y-homeomorphism by Lickorish as an example of an element of the mapping class group which cannot be expressed as a product of Dehn twists. We prove that the subgroup of the mapping class group of a closed nonorientable surface N generated by all crosscap slides is equal to the level 2 subgroup consisting of those mapping classes which act trivially on $H_1(N; \mathbb{Z}_2)$. We also prove that this subgroup is generated by involutions.

1. INTRODUCTION

It is well known that the mapping class group $\mathcal{M}(S)$ of a closed orientable surface S is generated by Dehn twists. This result was originally proved by Dehn [5] and rediscovered by Lickorish [10]. On a closed nonorientable surface N , however, there are elements of the mapping class group $\mathcal{M}(N)$ which cannot be expressed as a product of Dehn twists. The first example of such element is the crosscap slide, also called Y-homeomorphism, introduced by Lickorish [11]. He proved [11, 12] that the twist subgroup $\mathcal{T}(N)$, generated by Dehn twists about all two-sided simple closed curves, has index 2 in $\mathcal{M}(N)$, and that $\mathcal{M}(N)$ is generated by Dehn twists and one crosscap slide. Later, Chillingworth [4] found finite generating sets for $\mathcal{M}(N)$ and $\mathcal{T}(N)$. Korkmaz [8, 9] and Stukow [19, 21] extended Chillingworth's results to surfaces with punctures and boundary and also computed the abelianizations of $\mathcal{M}(N)$ and $\mathcal{T}(N)$.

In this paper we consider the subgroup $\mathcal{Y}(N)$ of $\mathcal{M}(N)$ generated by all crosscap slides. Our main result is Theorem 5.5, which asserts that for a closed nonorientable surface N of genus $g \geq 2$, $\mathcal{Y}(N)$ is equal to the level 2 subgroup $\Gamma_2(N)$ of $\mathcal{M}(N)$ consisting of those mapping classes which act trivially on $H_1(N; \mathbb{Z}_2)$. In particular $\mathcal{Y}(N)$ is a subgroup of finite index. We also prove that $\mathcal{Y}(N)$ is generated by involutions (Theorem 3.7).

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It is an interesting problem to find a finite generating set for $\mathcal{Y}(N)$ and also compute its abelianization. For an orientable surface S , the abelianization of the level 2 subgroup of $\mathcal{M}(S)$ was computed by Sato [17].

The paper is organised as follows. After Preliminaries, we prove in Section 3 that certain elements of $\mathcal{M}(N)$ belong to $\mathcal{Y}(N)$ and that the last group is generated by involutions. In Section 4 we study the level m mapping class groups $\Gamma_m(S)$ of an orientable surface S . We use known results about generators of Torelli groups and the Bass-Milnor-Serre's solution to the congruence subgroup problem for the symplectic group, to obtain convenient generating sets of $\Gamma_m(S)$. In Section 5 we use lemmas proved in previous sections, as well as some results of McCarthy-Pinkall [13] and Korkmaz [8], to prove our main result, which identifies the group $\mathcal{Y}(N)$ with the level 2 mapping class group $\Gamma_2(N)$.

Acknowledgment. I was informed by Professor Susumu Hirose that he found a shorter proof of my Theorem 5.5 using Nowik's result [14, Theorem 3.6]. I thank him for the discussion and for the reference [14].

2. PRELIMINARIES

Let F be a connected compact surface, possibly with boundary. Define $\text{Homeo}(F)$ to be the group of all, orientation preserving if F is orientable, homeomorphisms $h: F \rightarrow F$ equal to the identity on the boundary ∂F . The *mapping class group* $\mathcal{M}(F)$ is the group of isotopy classes in $\text{Homeo}(F)$. By abuse of notation we will use the same symbol to denote a homeomorphism and its isotopy class. If g and h are two homeomorphisms, then the composition gh means that h is applied first. A surface of genus g with n boundary components will be denoted by $S_{g,n}$ if it is orientable, or by $N_{g,n}$ if it is nonorientable. The genus of a nonorientable surface is the number of projective planes in the connected sum decomposition.

2.1. Simple closed curves and Dehn twists. By a *simple closed curve* in F we mean an embedding $\gamma: S^1 \rightarrow F \setminus \partial F$. Note that γ has an orientation; the curve with the opposite orientation but same image will be denoted by γ^{-1} . By abuse of notation, we will often identify a simple closed curve with its oriented image and also with its isotopy class. According to whether a regular neighborhood of γ is an annulus or a Möbius strip, we call γ respectively *two-* or *one-sided*. We say that γ is *nonseparating* if $F \setminus \gamma$ is connected and *separating* otherwise.

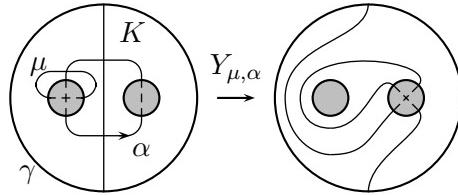


FIGURE 1. Crosscap slide.

If $\gamma = \partial M$ for some subsurface $M \subset F$ then we call γ *bounding*. If $n \leq 1$ then every separating curve is bounding.

Given a two-sided simple closed curve γ , T_γ denotes a Dehn twist about γ . On a nonorientable surface it is impossible to distinguish between right and left twists, so the direction of a twist T_γ has to be specified for each curve γ . Equivalently we may choose an orientation of a regular neighborhood of γ . Then T_γ denotes the right Dehn twist with respect to the chosen orientation. Unless we specify which of the two twists we mean, T_γ denotes any of the two possible twists. Recall that T_γ does not depend on the orientation of γ .

Bounding pair is a pair (γ, γ') of disjoint simple closed curves, such that $\gamma \cup \gamma' = \partial S$ for an orientable subsurface $S \subset F$. If $T_\gamma, T_{\gamma'}$ are right Dehn twists with respect to some orientation of S , then $T_\gamma T_{\gamma'}^{-1}$ is called *bounding pair map*. Our definition of a bounding pair differs slightly from the usual definition, in which the curves γ and γ' are required to be nonseparating.

2.2. Crosscap slides. Let $N = N_{g,n}$ be a nonorientable surface of genus $g \geq 2$. Suppose that μ and α are two simple closed curves in N , such that μ is one-sided, α is two-sided and they intersect in one point. Let $K \subset N$ be a regular neighborhood of $\mu \cup \alpha$, which is homeomorphic to the Klein bottle with a hole, and let $M \subset K$ be a regular neighborhood of μ , which is a Möbius strip. We denote by $Y_{\mu,\alpha}$ the *crosscap slide*, or *Y-homeomorphism* of K which may be described as the result of pushing M once along α keeping the boundary of K fixed. Figure 1 illustrates the effect of $Y_{\mu,\alpha}$ on an arc connecting two points in the boundary of K . Here, and also in other figures of this paper, the shaded discs represent crosscaps; this means that their interiors should be removed, and then antipodal points in each resulting boundary component should be identified. The homeomorphism $Y_{\mu,\alpha}$ pushes the left crosscap through the right one, along α . Observe that $Y_{\mu,\alpha}$ reverses the orientation of μ . We extend $Y_{\mu,\alpha}$ by the identity outside K to a homoeomorphism of N , which we denote by the same

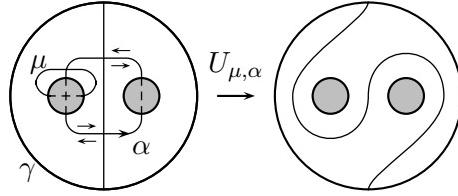


FIGURE 2. Crosscap transposition.

symbol. Up to isotopy, $Y_{\mu,\alpha}$ does not depend on the choice of the regular neighbourhood K . It also does not depend on the orientation of μ but does depend on the orientation of α . The following properties of crosscap slides are easy to verify.

$$(2.1) \quad Y_{\mu,\alpha^{-1}} = Y_{\mu,\alpha}^{-1},$$

$$(2.2) \quad Y_{\mu,\alpha}^2 = T_\gamma,$$

where T_γ is the Dehn twist about $\gamma = \partial K$, right with respect to the standard orientation of the plane of Figure 1,

$$(2.3) \quad hY_{\mu,\alpha}h^{-1} = Y_{h(\mu),h(\alpha)},$$

for all $h \in \text{Homeo}(N)$. The crosscap slide is an example of an element of $\mathcal{M}(N)$ which is not a product of Dehn twists. It was introduced by Lickorish, who proved that $\mathcal{M}(N_g)$ is generated by Dehn twists and one crosscap slide for $g \geq 2$ [11, 12].

Let T_α denote the Dehn twist about α in the direction indicated by the arrows in Figure 2. Then

$$U_{\mu,\alpha} = T_\alpha Y_{\mu,\alpha}$$

interchanges two crosscaps. We call this homeomorphism *crosscap transposition*.

2.3. Blowup homomorphism and crosscap pushing map. Let $N = N_{g,n}$ and fix $x_0 \in N \setminus \partial N$. Define $\mathcal{M}(N, x_0)$ to be the group of isotopy classes in $\text{Homeo}(N, x_0)$, the group of all homeomorphisms $h: N \rightarrow N$ equal to the identity on ∂N and such that $h(x_0) = x_0$.

Let $U = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and fix an embedding $e: U \rightarrow N \setminus \partial N$ such that $e(0) = x_0$. The surface $N_{g+1,n}$ may be obtained by removing from N the interior of $e(U)$ and then identifying $e(z)$ with $e(-z)$ for $z \in S^1 = \partial U$. Let $\gamma = e(S^1)$ and let μ be the image of γ in $N_{g+1,n}$. Note that μ is a one-sided simple closed curve.

We define a *blowup homomorphism*

$$\varphi: \mathcal{M}(N_{g,n}, x_0) \rightarrow \mathcal{M}(N_{g+1,n})$$

as follows. Represent $h \in \mathcal{M}(N, x_0)$ by a homeomorphism $h: N \rightarrow N$ such that (a) h is equal to the identity on $e(U)$ or (b) $h(x) = e(\overline{e^{-1}(x)})$ for $x \in e(U)$. Such h commutes with the identification leading to $N_{g+1,n}$ and thus induces an element $\varphi(h) \in \mathcal{M}(N_{g+1,n})$. To see that φ is well defined consider two homeomorphisms $h_1, h_2 \in \text{Homeo}(N, x_0)$ which satisfy (a) or (b) and are isotopic by an isotopy fixing x_0 . Let $N' = N \setminus \text{int}(e(U))$. By [20, Theorem 3.6] the kernel of the homomorphism $i_*: \mathcal{M}(N') \rightarrow \mathcal{M}(N, x_0)$ induced by the inclusion of N' in N is generated by T_γ . It follows that the restriction of $h_1 h_2^{-1}$ to N' is isotopic by an isotopy fixed on γ to some power of T_γ . Since a Dehn twist about a curve bounding a Möbius strip is trivial, h_1 and h_2 induce the same element of $\mathcal{M}(N_{g+1,n})$.

Forgetting the distinguished point x_0 induces a homomorphism

$$\mathcal{M}(N_{g,n}, x_0) \rightarrow \mathcal{M}(N_{g,n}),$$

which fits into the Birman exact sequence (see [9])

$$\pi_1(N_{g,n}, x_0) \xrightarrow{j} \mathcal{M}(N_{g,n}, x_0) \rightarrow \mathcal{M}(N_{g,n}) \rightarrow 1.$$

The homomorphism j is called *point pushing map*. If α is a loop in $N_{g,n}$ based at x_0 and $[\alpha] \in \pi_1(N_{g,n}, x_0)$ is its homotopy class, then $j([\alpha])$ may be described as the effect of pushing x_0 once along α .

We define a *crosscap pushing map* to be the composition

$$\psi = \varphi \circ j: \pi_1(N_{g,n}, x_0) \rightarrow \mathcal{M}(N_{g+1,n}).$$

The following two lemmas follow from the description of the point pushing map for nonorientable surfaces [9, Lemma 2.2 and Lemma 2.3] and the definition of a crosscap slide.

Lemma 2.1. *Suppose that α is a two-sided simple loop in $N_{g,n}$ based at x_0 and let M be a regular neighbourhood of α containing $e(U)$. Let M' be the image in $N_{g+1,n}$ of $M \setminus \text{int}(e(U))$. Then*

$$\psi([\alpha]) = T_\beta T_\delta^{-1},$$

where β and δ are boundary curves of M' and the twists are right with respect to some orientation of $M' \setminus \mu$.

Lemma 2.2. *Suppose that α is a one-sided simple loop in $N_{g,n}$ based at x_0 . Let α' be a simple loop in the homotopy class $[\alpha]$ which intersects γ in two antipodal points. Then*

$$\psi([\alpha]) = Y_{\mu, \tilde{\alpha}},$$

where $\tilde{\alpha}$ is the image in $N_{g+1,n}$ of $\alpha' \setminus \text{int}(e(U))$.

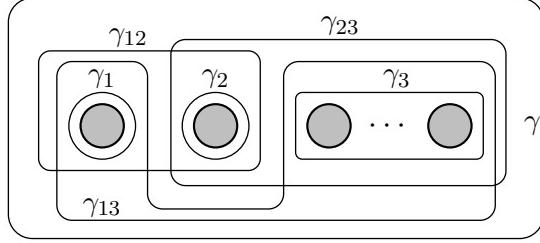


FIGURE 3. Curves of the lantern relation.

Corollary 2.3. *For $n \leq 1$ every element of $\psi(\pi_1(N_{g,n}, x_0))$ is a product of crosscap slides.*

Proof. For $n \leq 1$, $\pi_1(N_{g,n}, x_0)$ is generated by homotopy classes of one-sided simple loops. Since ψ is a homomorphism, it follows by Lemma 2.2 that $\psi(\pi_1(N_{g,n}, x_0))$ is generated by crosscap slides. \square

3. THE GROUP GENERATED BY CROSSCAP SLIDES.

In this section we assume that $N = N_g$ is a closed nonorientable surface of genus $g \geq 2$. We denote by $\mathcal{Y}(N)$ the subgroup of $\mathcal{M}(N)$ generated by all crosscap slides. Since by (2.3) a conjugate of a crosscap slide is also a crosscap slide, $\mathcal{Y}(N)$ is a normal subgroup of $\mathcal{M}(N)$.

Lemma 3.1. *For every nonseparating two-sided simple closed curve α in N , T_α^2 belongs to $\mathcal{Y}(N)$.*

Proof. Let μ be a one-sided curve intersecting α in one point. Since $Y_{\mu,\alpha}$ preserves the curve α and reverses orientation of its neighbourhood, we have $Y_{\mu,\alpha}T_\alpha Y_{\mu,\alpha}^{-1} = T_\alpha^{-1}$, and by (2.1, 2.3)

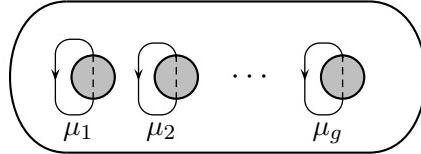
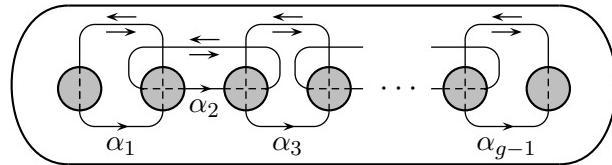
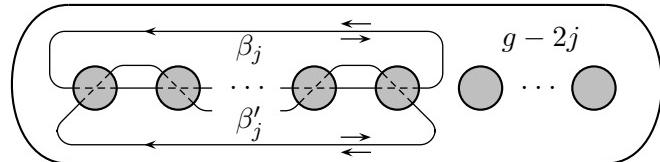
$$T_\alpha^2 = T_\alpha Y_{\mu,\alpha} T_\alpha^{-1} Y_{\mu,\alpha}^{-1} = Y_{T_\alpha(\mu),\alpha} Y_{\mu,\alpha^{-1}}.$$

 \square

Lemma 3.2. *For every separating simple closed curve γ in N , T_γ belongs to $\mathcal{Y}(N)$.*

Proof. Let K be a nonorientable subsurface of N such that $\gamma = \partial K$. We will argue by induction on the genus ρ of K . If $\rho = 1$ then T_γ is trivial, and if $\rho = 2$ then T_γ is the square of a crosscap slide supported in K by (2.2). Suppose that $\rho \geq 3$ and consider the separating curves in Figure 3. Note that $\gamma, \gamma_1, \gamma_2, \gamma_3$, bound a four-holed sphere and we have the well known lantern relation

$$T_\gamma T_{\gamma_1} T_{\gamma_2} T_{\gamma_3} = T_{\gamma_{12}} T_{\gamma_{13}} T_{\gamma_{23}},$$

FIGURE 4. The one-sided curves μ_i .FIGURE 5. The two-sided curves α_i .FIGURE 6. The pair (β_j, β'_j) bounding an orientable subsurface of genus $j - 1$.

where the twists are right with respect to the standard orientation of the plane of the figure. Since each of the curves different from γ bounds a nonorientable subsurface of genus less than ρ , the corresponding twist is in $\mathcal{Y}(N)$ by inductive hypothesis. It follows that $T_\gamma \in \mathcal{Y}(N)$. \square

Represent the surface $N = N_g$ ($g \geq 2$) as a 2-sphere with g cross-caps (Figure 4). Recall that it means that the interiors of the shaded disc should be removed and then antipodal points on each resulting boundary component should be identified. Assume that the sphere is embedded in \mathbb{R}^3 in such a way that it is invariant under the reflection about a plane which contains centers of the shaded discs. The reflection commutes with the identification leading to N , so it induces a homeomorphism $R: N \rightarrow N$ of order 2.

For $1 \leq i \leq g - 1$ let

$$Y_i = Y_{\mu_i, \alpha_i}, \quad U_i = U_{\mu_i, \alpha_i} = T_{\alpha_i} Y_{\mu_i, \alpha_i} = Y_{\mu_i, \alpha_i} T_{\alpha_i}^{-1},$$

where μ_i is the one-sided curve in Figure 4 and T_{α_i} is the Dehn twist about α_i in Figure 5 in the indicated direction. For $1 \leq j \leq \lfloor g/2 \rfloor$ let

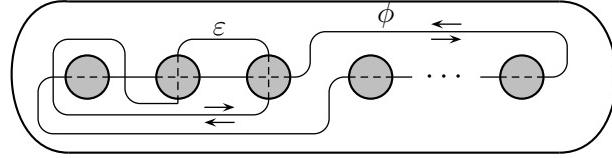


FIGURE 7. Curves from the proof of Lemma 3.5.

$T_{\beta_j}, T_{\beta'_j}$ be Dehn twists about the curves in Figure 6 in the indicated directions. Observe that β_j and β'_j bound an orientable subsurface of genus $j - 1$ and $T_{\beta_j}, T_{\beta'_j}$ are right twists with respect to appropriate orientation of this subsurface, hence $T_{\beta_j}T_{\beta'_j}^{-1}$ is a bounding pair map.

Theorem 3.3 (Chillingworth [4]). *The group $\mathcal{M}(N_g)$ is generated by*

$$\{T_{\alpha_i}, T_{\beta_j}, Y_{g-1}; 1 \leq i \leq g-1, 1 \leq j \leq \lfloor g/2 \rfloor\}.$$

Lemma 3.4. *The involution R belongs to $\mathcal{Y}(N)$.*

Proof. Let $F = RU_{g-1}U_{g-2} \cdots U_1T_{\alpha_1}T_{\alpha_2} \cdots T_{\alpha_{g-1}}$. It is straightforward to check that F fixes each of the curves μ_i, α_i, β_j and preserves orientation of regular neighbourhoods of the last two for $1 \leq i \leq g-1$, $1 \leq j \leq \lfloor g/2 \rfloor$. It follows that F commutes with $Y_i, T_{\alpha_i}, T_{\beta_j}$, and thus by Theorem 3.3 it belongs to the center of $\mathcal{M}(N_g)$ which is trivial, according to [18, Corollary 6.3]. Hence $F = 1$ and

$$\begin{aligned} R &= U_{g-1}U_{g-2} \cdots U_1T_{\alpha_1}T_{\alpha_2} \cdots T_{\alpha_{g-1}} \\ &= U_{g-1}T_{\alpha_{g-1}}T_{\alpha_{g-1}}^{-1}U_{g-2}T_{\alpha_{g-2}}T_{\alpha_{g-1}}T_{\alpha_{g-1}}^{-1}T_{\alpha_{g-2}}^{-1} \cdots U_1T_{\alpha_1}T_{\alpha_2} \cdots T_{\alpha_{g-1}} \\ &= Y_{g-1}T_{\alpha_{g-1}}^{-1}Y_{g-2}T_{\alpha_{g-1}} \cdots (T_{\alpha_2} \cdots T_{\alpha_{g-1}})^{-1}Y_1(T_{\alpha_2} \cdots T_{\alpha_{g-1}}) \in \mathcal{Y}(N). \end{aligned}$$

□

Let N_{g-1} be the surface obtained by replacing the leftmost crosscap in Figure 4 by a disc with center x_0 . Taking that disc as $e(U)$, N may be seen as being obtained from N_{g-1} by the blowup construction described in Subsection 2.3. As a result we have the crosscap pushing map

$$\psi: \pi_1(N_{g-1}, x_0) \rightarrow \mathcal{M}(N).$$

By Corollary 2.3 $\psi(\pi_1(N_{g-1}, x_0)) \subset \mathcal{Y}(N)$.

Lemma 3.5. *Every bounding pair map belongs to $\mathcal{Y}(N)$.*

Proof. Let $T_\gamma T_{\gamma'}^{-1}$ be a bounding pair map, where $\gamma \cup \gamma'$ is the boundary of an orientable subsurface S of genus $j - 1$ and the twists $T_\gamma, T_{\gamma'}$ are right with respect to some orientation of S . If γ and γ' are separating,

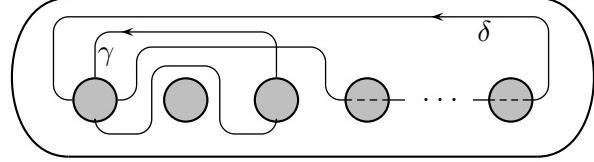


FIGURE 8. Curves from the proof of Lemma 3.6.

then we are done by Lemma 3.2, therefore we are assuming that γ and γ' are nonseparating.

If $N \setminus S$ is nonorientable, then there is a homeomorphism $h: N \rightarrow N$ which maps (γ, γ') to (β_j, β'_j) (Figure 6), so $hT_\gamma T_{\gamma'}^{-1} h^{-1} = (T_{\beta_j} T_{\beta'_j}^{-1})^{\pm 1}$ and it suffices to prove that $T_{\beta_j} T_{\beta'_j}^{-1} \in \mathcal{Y}(N)$. We have $RT_{\beta_j} R = T_{\beta'_j}$ and $T_{\beta_j} T_{\beta'_j}^{-1} = T_{\beta_j} RT_{\beta'_j}^{-1} R$, and since R is in $\mathcal{Y}(N)$ by Lemma 3.4, thus $T_{\beta_j} RT_{\beta'_j}^{-1} \in \mathcal{Y}(N)$ and $T_{\beta_j} T_{\beta'_j}^{-1} \in \mathcal{Y}(N)$.

Suppose that $N \setminus S$ is orientable. This can only happen if g is even, say $g = 2r$, and $N \setminus \gamma$ and $N \setminus \gamma'$ are orientable. We proceed by induction on j . If $j = 1$ then S is an annulus and $T_\gamma T_{\gamma'}^{-1} = 1$. If $j = 2$ then $T_\gamma T_{\gamma'}^{-1}$ is conjugate in $\mathcal{M}(N)$ to $(T_{\beta_r} T_\phi^{-1})^{\pm 1}$, where β_r is the curve in Figure 6 and ϕ is the curve in Figure 7. As in the previous case, it suffices to show that $T_\phi = fT_{\beta_r}f^{-1}$ for some $f \in \mathcal{Y}(N)$. It is easy to check that we may take $f = U_2^{-1}T_\varepsilon = Y_2^{-1}T_{\alpha_2}^{-1}T_\varepsilon$, where ε is the curve in Figure 7. To see that $T_{\alpha_2}^{-1}T_\varepsilon \in \mathcal{Y}(N)$ notice that ε and α_2 bound a Möbius strip with two holes, and by Lemma 2.1, $T_{\alpha_2}^{-1}T_\varepsilon$ is in the image of the crosscap pushing map ψ , which is a subgroup of $\mathcal{Y}(N)$.

If $j > 2$ then let γ'' be a separating simple closed curve in S such that $\gamma \cup \gamma'' = \partial S'$, $\gamma' \cup \gamma'' = \partial S''$ for subsurfaces S' , S'' of genera less than $j - 1$. We have $T_\gamma T_{\gamma'}^{-1} = (T_\gamma T_{\gamma''}^{-1})(T_{\gamma''} T_{\gamma'}^{-1}) \in \mathcal{Y}(N)$ because the bounding pair maps on the right hand side are in $\mathcal{Y}(N)$ by the inductive hypothesis. \square

Lemma 3.6. *Every crosscap slide is a product of crosscap slides conjugate in $\mathcal{M}(N)$ to Y_1 .*

Proof. Let $Y = Y_{\mu, \alpha}$ be any crosscap slide and let K and be a regular neighbourhood of $\alpha \cup \mu$, which is homeomorphic to the Klein bottle with a hole. Clearly there is a homeomorphism $h: K \rightarrow K_1$, where K_1 is a regular neighbourhood of $\alpha_1 \cup \mu_1$, such that $h(\alpha) = \alpha_1$ and $h(\mu) = \mu_1$. If $N \setminus K$ is nonorientable or $g = 2$, then h may be extended to $h: N \rightarrow N$ and by (2.3) we have $hY_{\mu, \alpha}h^{-1} = Y_{\mu_1, \alpha_1}$, hence Y is conjugate to Y_1 in $\mathcal{M}(N)$. Suppose that $g > 2$ and $N \setminus K$ is orientable.

This forces the genus of N to be even, $g = 2r \geq 4$. Now $Y_{\mu,\alpha}$ is conjugate in $\mathcal{M}(N)$ to Y_{μ_1,β_r} , where β_r is the curve in Figure 6 for $j = r$ ($N \setminus \beta_r$ is orientable). Note that Y_{μ_1,β_r} is in the image of the crosscap pushing map $\psi: \pi_1(N_{g-1}, x_0) \rightarrow \mathcal{M}(N)$. Consider the two-sided curves γ, δ in Figure 8. Since ψ is a homomorphism, sliding μ_1 along β_r is the same, up to isotopy, as sliding μ_1 along α_1 first, then along γ , then along δ . Thus $Y_{\mu_1,\beta_r} = Y_{\mu_1,\delta} Y_{\mu_1,\gamma} Y_{\mu_1,\alpha_1}$, where each of the three crosscap slides on the right hand side is conjugate to Y_1 by the previous argument. \square

Theorem 3.7. *The group $\mathcal{Y}(N)$ is generated by involutions.*

Proof. Since $\mathcal{Y}(N)$ is a normal subgroup of $\mathcal{M}(N)$ and conjugate of an involution is also an involution, it suffices, by Lemma 3.6, to show that Y_1 is a product of involutions in $\mathcal{Y}(N)$. The reflectional involution R is in $\mathcal{Y}(N)$ by Lemma 3.4. Since R maps μ_1, α_1 to $\mu_1^{-1}, \alpha_1^{-1}$, we have $RY_1R = Y_1^{-1}$, thus RY_1 is an involution in $\mathcal{Y}(N)$ and Y_1 is a product of two involutions $Y_1 = R(RY_1)$. \square

4. LEVEL m MAPPING CLASS GROUPS

In this section we assume that $S = S_{g,n}$ is an orientable surface of genus $g \geq 2$ having $n \in \{0, 1\}$ boundary components.

The action of $\mathcal{M}(S)$ on $H_1(S; \mathbb{Z})$ preserves the algebraic intersection pairing, which is a nondegenerate alternating form, so we have a representation of $\mathcal{M}(S)$ into the symplectic group, which is well known to be surjective and whose kernel $\mathcal{I}(S)$ is known as the *Torelli group*. This is summarized by the exact sequence

$$1 \longrightarrow \mathcal{I}(S) \longrightarrow \mathcal{M}(S) \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1.$$

We are going to use the following classical theorem proved by Powell [15] building on Birman's result [2].

Theorem 4.1 (Powell [15]). *The Torelli group $\mathcal{I}(S)$ is generated by Dehn twists about separating curves and bounding pair maps.*

The result above was later improved by Johnson [6] who showed, that for $g \geq 3$, $\mathcal{I}(S)$ is generated by finitely many bounding pair maps.

For $m \geq 2$, let $\mathrm{Sp}(2g, \mathbb{Z})[m]$ denote the *level m congruence subgroup* of $\mathrm{Sp}(2g, \mathbb{Z})$, that is, the subgroup of matrices that are equal to the identity modulo m . There is an exact sequence

$$1 \longrightarrow \mathrm{Sp}(2g, \mathbb{Z})[m] \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}_m) \longrightarrow 1,$$

where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. The pull-back of $\mathrm{Sp}(2g, \mathbb{Z})[m]$ to $\mathcal{M}(S)$ is known as the *level m subgroup* of $\mathcal{M}(S)$ and is denoted by $\Gamma_m(S)$. The group

$\Gamma_m(S)$ can also be described as the group of isotopy classes of homeomorphisms that act trivially on $H_1(S; \mathbb{Z}_m)$. It fits into an exact sequence

$$(4.1) \quad 1 \longrightarrow \mathcal{I}(S) \longrightarrow \Gamma_m(S) \longrightarrow \mathrm{Sp}(2g, \mathbb{Z})[m] \longrightarrow 1.$$

It is well known that for $m \geq 3$, $\mathrm{Sp}(2g, \mathbb{Z})[m]$ and $\Gamma_m(S)$ are torsion free. The case $m = 2$ is exceptional, $\Gamma_2(S_{g,0})$ contains torsion elements, for example the hyperelliptic involution.

For $1 \leq i, j \leq g$ let $C_{ij}^g(m)$ be the $g \times g$ matrix with m at positions (i, j) and (j, i) and 0 elsewhere. Denoting the $g \times g$ zero matrix by O_g and the $g \times g$ identity matrix by I_g , let

$$A_{ij}^g(m) = \begin{pmatrix} I_g & C_{i,j}^g(m) \\ O_g & I_g \end{pmatrix}, \quad B_{ij}^g(m) = \begin{pmatrix} I_g & O_g \\ C_{i,j}^g(m) & I_g \end{pmatrix}.$$

The following theorem forms part of Bass-Milnor-Serre's solution to the congruence subgroup problem for the symplectic group.

Theorem 4.2 (Bass-Milnor-Serre [1, Theorem 12.4, Corollary 12.5]). *For $g \geq 2$ the group $\mathrm{Sp}(2g, \mathbb{Z})[m]$ is generated by $\{A_{ij}^g(m), B_{ij}^g(m); 1 \leq i, j \leq g\}$.*

For $h_1, h_2 \in H_1(S; \mathbb{Z})$ let $i(h_1, h_2)$ denote the algebraic intersection number. Let us fix a symplectic basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ of $H_1(S; \mathbb{Z})$, that is

$$i(a_i, b_j) = \delta_{ij}, \quad i(a_i, a_j) = 0, \quad i(b_i, b_j) = 0,$$

for $1 \leq i, j \leq g$, where δ_{ij} is the Kronecker delta. We also fix a realisation of this basis by simple closed curves $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$, such that $a_i = [\alpha_i]$, $b_i = [\beta_i]$ and

$$|\alpha_i \cap \beta_j| = \delta_{ij}, \quad \alpha_i \cap \alpha_j = \emptyset, \quad \beta_i \cap \beta_j = \emptyset,$$

for $1 \leq i, j \leq g$. Recall that a homology class $h \in H_1(S; \mathbb{Z})$ can be realised by a simple closed curve if and only if h is primitive, which means that the greatest common divisor of the coefficients of h in a symplectic basis equals 1. For $f \in \mathcal{M}(S)$ we denote by \overline{f} the induced automorphism of $H_1(S; \mathbb{Z})$ and, by abuse of notation, its matrix with respect to the basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$. If γ is a simple closed curve on S , then the right Dehn twist T_γ acts on $H_1(S; \mathbb{Z})$ by the transvection

$$\overline{T_\gamma}(h) = h + i(h, [\gamma])[\gamma].$$

Lemma 4.3. *Let $S = S_{g,n}$ for $g \geq 2$ and $n \in \{0, 1\}$. Every element of $\mathrm{Sp}(2g, \mathbb{Z})[m]$ is induced by a product of m -th powers of Dehn twists about nonseparating curves in S .*

Proof. By Theorem 4.2 it suffices to show that $A_{ij}^g(m)$ and $B_{ij}^g(m)$ are induced by products of m -th powers of Dehn twists about non-separating curves in S for $1 \leq i, j \leq g$. We have $\overline{T_{\alpha_i}^{-m}} = A_{ii}^g(m)$ and $\overline{T_{\beta_i}^m} = B_{ii}^g(m)$ for $1 \leq i \leq g$. For $i \neq j$ let γ_{ij} , δ_{ij} be simple closed curves such that $[\gamma_{ij}] = a_i + a_j$, $[\delta_{ij}] = b_i + b_j$. Then $\overline{T_{\alpha_i}^m T_{\alpha_j}^m T_{\gamma_{ij}}^{-m}} = A_{ij}^g(m)$, $\overline{T_{\beta_i}^{-m} T_{\beta_j}^{-m} T_{\delta_{ij}}^m} = B_{ij}^g(m)$. \square

From the exactness of sequence (4.1), Theorem 4.1 and Lemma 4.3 we have the following.

Corollary 4.4. *Let $S = S_{g,n}$ for $g \geq 2$ and $n \in \{0, 1\}$. The group $\Gamma_m(S)$ is generated by Dehn twists about separating curves, bounding pair maps and m -th powers of Dehn twists about nonseparating curves.*

For the rest of this section assume that $S = S_g$ is closed and $\Sigma = S_{g-1,2}$. Consider an embedding $i: \Sigma \hookrightarrow S$ such that $S \setminus i(\Sigma)$ is an annulus with core β_g . Such embedding is called cupping in [16]. Since every homeomorphism of $i(\Sigma)$ equal to the identity on $i(\partial\Sigma)$ can be extended by the identity on $S \setminus i(\Sigma)$ to a homeomorphism of S , we have an induced homomorphism $i_*: \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(S)$. The image of i_* is equal to the subgroup $\text{Stab}_{\mathcal{M}(S)}(\beta_g)$ consisting of the isotopy classes of homeomorphisms fixing the curve β_g and preserving its orientation. Let δ_1, δ_2 be two simple closed curves in Σ isotopic to the boundary components. The kernel of i_* is generated by the bounding pair map $T_{\delta_1} T_{\delta_2}^{-1}$. Summarising, we have an exact sequence

$$1 \longrightarrow \langle T_{\delta_1} T_{\delta_2}^{-1} \rangle \longrightarrow \mathcal{M}(\Sigma) \longrightarrow \text{Stab}_{\mathcal{M}(S)}(\beta_g) \longrightarrow 1.$$

Following [16] we let

$$\mathcal{I}(\Sigma) = i_*^{-1}(\mathcal{I}(S)).$$

For a surface having $n > 1$ boundary components various definitions of a Torelli group are possible. The group $\mathcal{I}(\Sigma)$ defined above is the same as $\mathcal{I}(\Sigma, \{\{\delta_1, \delta_2\}\})$ in the notation of [16] and it agrees with the definition of the Torelli group of a surface with boundary given by Johnson [7]. Note that $\mathcal{I}(\Sigma)$ is not the kernel of the action of $\mathcal{M}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$. For example, T_{δ_i} for $i = 1, 2$ act trivially on the homology group, but they are not elements of $\mathcal{I}(\Sigma)$. Nevertheless, $\mathcal{I}(\Sigma)$ can also be defined intrinsically, without reference to S , by using relative homology, see [7, 16]. Note that γ is a bounding curve in Σ if and only if $i(\gamma)$ is separating in S , and (γ, γ') is a bounding pair in Σ if and only if $(i(\gamma), i(\gamma'))$ is a bounding pair in S .

Theorem 4.5 (Putman [16]). *The group $\mathcal{I}(\Sigma)$ is generated by Dehn twists about bounding curves and bounding pair maps.*

Let $\text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g)$ be the subgroup of $\text{Sp}(2g, \mathbb{Z})$ consisting of the (matrices of) automorphisms of $H_1(S; \mathbb{Z})$ which fix the homology class $b_g = [\beta_g]$, and

$$\text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g) = \text{Sp}(2g, \mathbb{Z})[m] \cap \text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g).$$

Observe that we have an isomorphism

$$\text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g)/\text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g) \cong \text{Stab}_{\text{Sp}(2g, \mathbb{Z}_m)}(\overline{b_g}),$$

where $\overline{b_g}$ is the homology class in $H_1(S; \mathbb{Z}_m)$ of the curve β_g .

Lemma 4.6. *For $g \geq 3$ every element of $\text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g)$ is induced by a product of m -th powers of Dehn twists about nonseparating curves in S disjoint from β_g .*

Proof. Let N be a regular neighbourhood of $\alpha_g \cup \beta_g$ and $S' = S \setminus N$. Note that S' is a copy of $S_{g-1,1}$ and we have a symplectic splitting

$$H_1(S; \mathbb{Z}) = H_1(N; \mathbb{Z}) \oplus H_1(S'; \mathbb{Z}),$$

where $H_1(N; \mathbb{Z}) = \langle a_g, b_g \rangle$. Let $A \in \text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g)$ and consider $h = Aa_g$. We have

$$i(h, b_g) = i(Aa_g, Ab_g) = i(a_g, b_g) = 1,$$

thus

$$h = a_g + m(xb_g + h'),$$

where $x \in \mathbb{Z}$, $h' \in H_1(S'; \mathbb{Z})$. Let h'' be a primitive element of $H_1(S'; \mathbb{Z})$ such that $h' = yh''$ for $y \in \mathbb{Z}$ (if $h' = 0$, then $y = 0$ and h'' is arbitrary) and let γ be any nonseparating simple closed curve in S' such that $h'' = [\gamma]$. Let δ be a simple closed curve disjoint from β_g and such that $[\delta] = [\gamma] - [\beta_g]$ (Figure 9). We have $\overline{T_{\beta_g}^{m(x+y)} T_{\delta}^{my}}(h) = a_g$. Since $\overline{T_{\beta_g}^{m(x+y)} T_{\delta}^{my}} A$ is equal to the identity on $\langle a_g, b_g \rangle$, thus it restricts to a symplectic automorphism of $H_1(S'; \mathbb{Z})$, and hence, by Lemma 4.3, it is induced by a product of m -th powers of Dehn twists about nonseparating curves in S' (here we use the assumption $g \geq 3$). Since T_{β_g} can be seen as a twist about a curve isotopic to β_g and disjoint from β_g , A is induced by a product of m -th powers of Dehn twists about nonseparating curves in S disjoint from β_g . \square

We define level m subgroup of $\mathcal{M}(\Sigma)$ as

$$\Gamma_m(\Sigma) = i_*^{-1}(\Gamma_m(S)).$$

Consider the map $\Gamma_m(\Sigma) \rightarrow \text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g)$ given by $f \mapsto \overline{i_*(f)}$. By Lemma 4.6 this map is surjective, because every simple closed curve

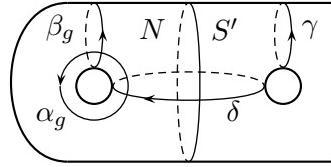


FIGURE 9. Curves from the proof of Lemma 4.6.

in S disjoint from β_g is isotopic to a curve in $i(\Sigma)$. We have an exact sequence

$$1 \longrightarrow \mathcal{I}(\Sigma) \longrightarrow \Gamma_m(\Sigma) \longrightarrow \text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g) \longrightarrow 1,$$

which together with Lemma 4.6 and Theorem 4.5 gives the following.

Corollary 4.7. *Let $g \geq 3$ and $\Sigma = S_{g-1,2}$. The group $\Gamma_m(\Sigma)$ is generated by Dehn twists about bounding curves, bounding pair maps and m -th powers of Dehn twists about nonseparating curves.*

Proposition 4.8. *The quotient group $\mathcal{M}(\Sigma)/\Gamma_m(\Sigma)$ is isomorphic to the stabiliser $\text{Stab}_{\text{Sp}(2g, \mathbb{Z}_m)}(\overline{b_g})$.*

Proof. First we show that the map $\text{Stab}_{\mathcal{M}(\Sigma)}(\beta_g) \rightarrow \text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g)$ is surjective. Let $A \in \text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g)$ and let $f \in \mathcal{M}(\Sigma)$ be such that $\overline{f} = A$. Since $[f(\beta_g)] = [\beta_g]$, there is $h \in \mathcal{I}(\Sigma)$ such that $hf(\beta_g) = \beta_g$ (see Lemma 6.2 in [16]). Now $hf \in \text{Stab}_{\mathcal{M}(\Sigma)}(\beta_g)$ and $\overline{hf} = A$. We have

$$\text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g) \cong \frac{\text{Stab}_{\mathcal{M}(\Sigma)}(\beta_g)}{\mathcal{I}(\Sigma) \cap \text{Stab}_{\mathcal{M}(\Sigma)}(\beta_g)} \cong \frac{\mathcal{M}(\Sigma)}{\mathcal{I}(\Sigma)},$$

where the last isomorphism is induced by $i_*: \mathcal{M}(\Sigma) \rightarrow \text{Stab}_{\mathcal{M}(\Sigma)}(\beta_g)$. We have a series of isomorphisms

$$\frac{\mathcal{M}(\Sigma)}{\Gamma_m(\Sigma)} \cong \frac{\mathcal{M}(\Sigma)/\mathcal{I}(\Sigma)}{\Gamma_m(\Sigma)/\mathcal{I}(\Sigma)} \cong \frac{\text{Stab}_{\text{Sp}(2g, \mathbb{Z})}(b_g)}{\text{Stab}_{\text{Sp}(2g, \mathbb{Z})[m]}(b_g)} \cong \text{Stab}_{\text{Sp}(2g, \mathbb{Z}_m)}(\overline{b_g}).$$

□

5. THE ACTION OF $\mathcal{M}(N)$ ON $H_1(N; \mathbb{Z}_2)$.

In this section we assume that $N = N_g$ is a closed nonorientable surface of genus $g \geq 2$. Recall that $H_1(N; \mathbb{Z})$ is generated by $\{c_1, \dots, c_g\}$, where $c_i = [\mu_i]$ for $1 \leq i \leq g$ are the homology classes of the one-sided curves μ_i (Figure 4). As a \mathbb{Z} -module it has the following presentation

$$(5.1) \quad H_1(N; \mathbb{Z}) = \langle c_1, \dots, c_g \mid 2(c_1 + \dots + c_g) = 0 \rangle.$$

The \mathbb{Z}_2 -homology classes $\overline{c_i}$ of the curves μ_i form a basis of $H_1(N; \mathbb{Z}_2)$, which is a vector space over \mathbb{Z}_2 of dimension g . The \mathbb{Z}_2 -valued intersection pairing is a symmetric bilinear form \langle , \rangle on $H_1(N; \mathbb{Z}_2)$ satisfying $\langle \overline{c_i}, \overline{c_j} \rangle = \delta_{ij}$ for $1 \leq i, j \leq g$. The following three theorems were proved by McCarthy and Pinkall [13].

Theorem 5.1 ([13, Theorem 1]). *Every automorphism of $H_1(N; \mathbb{Z})$ which preserves the \mathbb{Z}_2 -valued intersection pairing is induced by a homeomorphism of N .*

Theorem 5.2 ([13, Theorem 2]). *Every automorphism of $H_1(N; \mathbb{Z}_2)$ which preserves the \mathbb{Z}_2 -valued intersection pairing is induced by a homeomorphism of N .*

Theorem 5.3 ([13, Theorem 3]). *Every automorphism of $H_1(N; \mathbb{Z})$ inducing a trivial automorphism of $H_1(N; \mathbb{Z}_2)$ is induced by a homeomorphism of N which is a product of crosscap slides.*

By Theorem 5.2, the natural map

$$\mathcal{M}(N) \longrightarrow \text{Iso}(H_1(N; \mathbb{Z}_2))$$

is surjective, where $\text{Iso}(H_1(N; \mathbb{Z}_2))$ is the group of automorphisms of $H_1(N; \mathbb{Z}_2)$ which preserve \langle , \rangle .

Recall that a symmetric bilinear form Ω on a vector space V over a field of characteristic 2 is called symplectic if it is nondegenerate and $\Omega(v, v) = 0$ for all $v \in V$. On a closed orientable surface S_r , the \mathbb{Z}_2 -valued intersection pairing is a symplectic form on $H_1(S_r, \mathbb{Z}_2)$. Let us fix a symplectic basis $\{\overline{a_1}, \dots, \overline{a_r}, \overline{b_1}, \dots, \overline{b_r}\}$ of $H_1(S_r, \mathbb{Z}_2)$. We shall identify the group $\text{Sp}(2r, \mathbb{Z}_2)$ with the group of symplectic automorphisms of $H_1(S_r, \mathbb{Z}_2)$. A proof of the following proposition can be found in [8, Section 3].

Proposition 5.4. *For $r \geq 1$, the group $\text{Iso}(H_1(N_{2r+1}, \mathbb{Z}_2))$ is isomorphic to the symplectic group $\text{Sp}(2r, \mathbb{Z}_2)$, and its order is given by the formula*

$$|\text{Iso}(H_1(N_{2r+1}, \mathbb{Z}_2))| = 2^{r^2} \prod_{i=1}^r (2^{2i} - 1),$$

the group $\text{Iso}(H_1(N_{2r}, \mathbb{Z}_2))$ is isomorphic to the stabiliser $\text{Stab}_{\text{Sp}(2r, \mathbb{Z}_2)}(\overline{b_r})$, and its order is given by the formula

$$|\text{Iso}(H_1(N_{2r}, \mathbb{Z}_2))| = 2^{r^2} \prod_{i=1}^{r-1} (2^{2i} - 1).$$

We define level 2 subgroup $\Gamma_2(N)$ of $\mathcal{M}(N)$ to be the group of isotopy classes of homeomorphisms which act trivially on $H_1(N; \mathbb{Z}_2)$. If fits into an exact sequence

$$1 \longrightarrow \Gamma_2(N) \longrightarrow \mathcal{M}(N) \longrightarrow \text{Iso}(H_1(N; \mathbb{Z}_2)) \longrightarrow 1.$$

The following theorem is an improvement of Theorem 5.3. It is the main result of this paper.

Theorem 5.5. *A mapping class $f \in \mathcal{M}(N)$ acts trivially on $H_1(N; \mathbb{Z}_2)$ if and only if f is a product of crosscap slides.*

In other words $\Gamma_2(N) = \mathcal{Y}(N)$.

Proof. The crosscap slide $Y_1 = Y_{\mu_1, \alpha_1}$ acts on $H_1(N; \mathbb{Z})$ by

$$\overline{Y_1}(c_1) = -c_1, \quad \overline{Y_1}(c_2) = 2c_1 + c_2, \quad \overline{Y_1}(c_i) = c_i \text{ for } 3 \leq i \leq g.$$

It follows that Y_1 acts trivially on $H_1(N; \mathbb{Z}_2)$ and hence, by Lemma 3.6, $\mathcal{Y}(N) \subseteq \Gamma_2(N)$. For the equality it suffices to prove $[\mathcal{M}(N): \mathcal{Y}(N)] \leq [\mathcal{M}(N): \Gamma_2(N)]$. The index $[\mathcal{M}(N): \Gamma_2(N)]$ is equal to the order of $\text{Iso}(H_1(N; \mathbb{Z}_2))$, which is given in Proposition 5.4.

If $g = 2$, then $\mathcal{M}(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\Gamma_2(N) = \mathcal{Y}(N) \cong \mathbb{Z}_2$ (see [11]).

If $g = 3$, then $\mathcal{M}(N) \cong \text{GL}(2, \mathbb{Z})$ and it admits the presentation

$$\langle a, b, y \mid aba = bab, y^2 = 1, yay = a^{-1}, yby = b^{-1}, (aba)^4 = 1 \rangle,$$

where $a = T_{\alpha_1}$, $b = T_{\alpha_2}$, $y = Y_1$ (see [3]). By Lemma 3.6, a presentation for the quotient group $\mathcal{M}(N)/\mathcal{Y}(N)$ may be obtained by adding the relation $y = 1$ to the presentation of $\mathcal{M}(N)$. Thus

$$\mathcal{M}(N)/\mathcal{Y}(N) \cong \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle \cong \Sigma_3,$$

where Σ_n is the symmetric group on n letters. Since $[\mathcal{M}(N): \Gamma_2(N)] = 6$ by Proposition 5.4, thus $\mathcal{Y}(N) = \Gamma_2(N)$.

If $g = 4$, then we can use the presentation of $\mathcal{M}(N)$ given in [22] to obtain the following presentation for $\mathcal{M}(N)/\mathcal{Y}(N)$.

$$\begin{aligned} \mathcal{M}(N)/\mathcal{Y}(N) \cong & \langle b \mid b^2 = 1 \rangle \times \\ & \langle a_1, a_2, a_3 \mid a_1^2 = a_2^2 = a_3^2 = (a_1 a_2)^3 = (a_2 a_3)^3 = 1, a_1 a_3 = a_3 a_1 \rangle, \end{aligned}$$

where $b = \widetilde{T}_{\beta_2}$, $a_i = \widetilde{T}_{\alpha_i}$ for $i = 1, 2, 3$, where \widetilde{f} is the coset of $f \in \mathcal{M}(N)$ in $\mathcal{M}(N)/\mathcal{Y}(N)$. It follows that $\mathcal{M}(N)/\mathcal{Y}(N) \cong \mathbb{Z}_2 \times \Sigma_4$, and since $[\mathcal{M}(N): \Gamma_2(N)] = 48$ by Proposition 5.4, thus $\mathcal{Y}(N) = \Gamma_2(N)$.

Now we assume that $g = 2r + 1$ for $r \geq 2$. Let μ be a one-sided simple closed curve in N such that $N \setminus \mu$ is orientable, and which is disjoint from the curves α_i, β_j , $1 \leq i \leq g - 1$, $1 \leq j \leq r$ in Figures 5 and 6. Let M be a regular neighbourhood of μ , which is a

Möbius band, and $S = N \setminus M$. Note that S is a copy of $S_{r,1}$. Consider a homomorphism $j: \mathcal{M}(S) \rightarrow \mathcal{M}(N)/\mathcal{Y}(N)$ defined as $j = p \circ i_*$, where $i_*: \mathcal{M}(S) \rightarrow \mathcal{M}(N)$ is induced by the inclusion $S \hookrightarrow N$ and $p: \mathcal{M}(N) \rightarrow \mathcal{M}(N)/\mathcal{Y}(N)$ is the canonical projection. By Theorem 3.3, $\mathcal{M}(N)$ is generated by Y_{g-1} and Dehn twists about curves in S . It follows that j is surjective and hence

$$[\mathcal{M}(N): \mathcal{Y}(N)] = [\mathcal{M}(S): \ker j].$$

By Corollary 4.4, $\Gamma_2(S)$ is generated by Dehn twists about separating curves, bounding pair maps and squares of Dehn twists about nonseparating curves. It follows by Lemmas 3.1, 3.2 and 3.5 that $i_*(\Gamma_2(S)) \subseteq \mathcal{Y}(N)$, hence $\Gamma_2(S) \subseteq \ker j$. We have

$$[\mathcal{M}(S): \ker j] \leq [\mathcal{M}(S): \Gamma_2(S)] = |\mathrm{Sp}(2r, \mathbb{Z}_2)|.$$

Since $[\mathcal{M}(N): \Gamma_2(N)] = |\mathrm{Sp}(2r, \mathbb{Z}_2)|$ by Proposition 5.4, thus $\mathcal{Y}(N) = \Gamma_2(N)$.

Finally we assume that $g = 2r$ for $r \geq 3$. Let A be regular neighbourhood of the two-sided curve β_r and $\Sigma = N \setminus A$. Note that Σ is a copy of $S_{r-1,2}$. The proof is analogous to the case of odd genus. The homomorphism $j: \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(N)/\mathcal{Y}(N)$ defined as $j = p \circ i_*$, where $i_*: \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(N)$ is induced by the inclusion $\Sigma \hookrightarrow N$, is surjective by Theorem 3.3. By Corollary 4.6 and Lemmas 3.1, 3.2 and 3.5 we have $\Gamma_2(\Sigma) \subseteq \ker j$ and thus

$$[\mathcal{M}(N): \mathcal{Y}(N)] = [\mathcal{M}(\Sigma): \ker j] \leq [\mathcal{M}(\Sigma): \Gamma_2(\Sigma)] = |\mathrm{Stab}_{\mathrm{Sp}(2r, \mathbb{Z}_2)}(\overline{b_r})|,$$

the last equality following from Proposition 4.8. By Proposition 5.4 we have

$$[\mathcal{M}(N): \Gamma_2(N)] = |\mathrm{Stab}_{\mathrm{Sp}(2r, \mathbb{Z}_2)}(\overline{b_r})|,$$

and hence $\mathcal{Y}(N) = \Gamma_2(N)$. \square

Remark 5.6. Theorem 5.5 provides a characterisation of the elements of $\mathcal{Y}(N)$ in terms of the action on homology. Such characterisation is known also for the twist subgroup. Namely, a mapping class $f \in \mathcal{M}(N)$ is a product of Dehn twists about two-sided curves if and only if the determinant of the induced map $f_*: H_1(N; \mathbb{R}) \rightarrow H_1(N; \mathbb{R})$ equals 1 (see [19, Corollary 6.3]).

The following corollary is a consequence of Theorems 3.7 and 5.5.

Corollary 5.7. *The group $\Gamma_2(N)$ is generated by involutions.*

Consider the quotient $R_g = H_1(N_g, \mathbb{Z}) / \langle c \rangle$, where $c = c_1 + \dots + c_g$ is the unique homology class of order 2. It is immediate from the presentation 5.1 that R_g is the free \mathbb{Z} -module with basis given by the images

of c_1, \dots, c_{g-1} in R_g . Every automorphism of $H_1(N_g, \mathbb{Z})$ preserves c , and thus induces an automorphism of R_g . It was shown in [13, Section 2], that the group of automorphisms of $H_1(N_g, \mathbb{Z})$ which act trivially on $H_1(N_g, \mathbb{Z}_2)$ is isomorphic to the full group of automorphisms of R_g which act trivially on $R_g \otimes \mathbb{Z}_2$. Consequently, we have a surjection

$$\Gamma_2(N_g) \longrightarrow \mathrm{GL}(g-1, \mathbb{Z})[2],$$

where $\mathrm{GL}(n, \mathbb{Z})[2]$ is the level 2 congruence subgroup of $\mathrm{GL}(n, \mathbb{Z})$. The following corollary is probably well known to algebraists.

Corollary 5.8. *The group $\mathrm{GL}(n, \mathbb{Z})[2]$ is generated by involutions.*

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